DAMPING OF NORMAL DISTURBANCES OF A VISCOUS SPHERE

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A considerable number of reports have been devoted to the study of the problem of the normal oscillations of a viscous capillary liquid partially or fully bounded by a free surface. The results of the majority of them are presented in [1]. The question of the asymptotic form of solutions of the problem for large and small values of the viscosity ν, when all the other parameters are fixed, has been analyzed in particular. In [2] it was shown that in the case of stability for highly viscous, heavy liquids the rate of damping the disturbances can be as low as desired. The hypothesis, confirmed by the examination of concrete examples, that in the case of stability the rate of damping of oscillations also becomes as low as desired as the viscosity approaches zero was advanced in [1]. If this is true, then for each concrete problem one can find a value ν^* at which disturbances in the liquid die out most rapidly. In the present work we find the value ν^* for the classical problem of normal oscillations of a viscous sphere.

The problem of linear oscillations of a weightless liquid sphere with allowance for viscous and capillary forces is analyzed in the spherical coordinate system $\hat{r}$, θ, φ (the origin of coordinates coincides with the center of the sphere).

Let R be the radius of the sphere, ρ the liquid density, ν the kinematic viscosity, and σ the coefficient of surface tension. It is convenient to introduce dimensionless variables, choosing the quantities R, $\nu^{-1}R^2$, νR^{-1}, and $\rho\nu R^{-2}$ as the characteristic size, time, velocity, and pressure, respectively, and designating the dimensionless velocity vector, pressure, and departure of the free surface from the equilibrium shape along the normal to it as $\hat{u}(r, \theta, \varphi, t)$, $\hat{p}(r, \theta, \varphi, t)$, and $\hat{N}(\theta, \varphi, t)$, respectively, and $r = \hat{r}R^{-1}$.

Let all the unknown quantities depend on time through an $e^{-\lambda_p t}$ law, i.e.,

$$\hat{N}(\theta, \varphi, t) = N(\theta, \varphi)\, e^{-\lambda_p t}, \quad \hat{u}(r, \theta, \varphi, t) = u(r, \theta, \varphi)\, e^{-\lambda_p t},$$
$$\hat{p}(r, \theta, \varphi, t) = p(r, \theta, \varphi)\, e^{-\lambda_p t}.$$

Starting from the Stokes system and the linearized conditions at the free boundary, after separation of the time factor $e^{-\lambda_p t}$ we can obtain the system of equations

$$\Delta u + \lambda u = \nabla p, \ \operatorname{div} u = 0 \ \text{for} \quad r < 1, \tag{1}$$

$$\begin{gathered} \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta = 0, \quad \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{1}{r}u_\varphi = 0, \\ p - 2\frac{\partial u_r}{\partial r} + \alpha^2(2N + \Delta_\Gamma N) = 0, \quad u_r = -\lambda N \quad \text{for} \quad r = 1, \end{gathered} \tag{2}$$

where Δ_Γ is the Laplace–Beltrami operator on a sphere,

$$\alpha^2 = \sigma R \rho^{-1} \nu^{-2}; \ \lambda = \lambda_p \nu^{-1} R^2.$$

Some results of investigations [3-5] of the problem (1), (2) are briefly formulated below.

The eigenvalue λ must satisfy one of the equations

$$F_l(\lambda) = (l - 1)\lambda^{-1}; \tag{3}$$

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$$2F_l(\lambda) = \Phi_l(\lambda, \alpha) \equiv \frac{\lambda^2 - 2\beta_l\lambda + \alpha^2\omega_l}{\lambda^2 - 2\omega_l\lambda + \alpha^2\omega_l}, \tag{4}$$

where l is a natural number,

$$\omega_l = l(l-1)(l+2),\ \beta_l = (l-1)(2l+1),$$
$$F_l(\lambda) = \frac{1}{\sqrt{\lambda}} \frac{J_{l+3/2}(\sqrt{\lambda})}{J_{l+1/2}(\sqrt{\lambda})} = 2\sum_{q=1}^{\infty} \frac{1}{k^2_{l+1/2,q} - \lambda}, \tag{5}$$

where $k_{l+1/2,q}$ is the q-th root of the Bessel function $J_{l+1/2}(\lambda)$.

Each simple root λ_l corresponds to $2l + 1$ forms of oscillations, in which the free surface has the shape

$$N(\varphi, \theta) = N_{l,n} Y_l^n(\theta, \varphi), \quad n = 0, \pm 1, \ldots, \pm l,$$

where Y_l^n is a spherical harmonic.

The solutions of Eq. (4) with $l = 1$ and of Eq. (3) with any l correspond to disturbances not deforming the free boundary. Everywhere below $l \geq 2$.

It was shown [1] that for each $l \geq 2$ and $\alpha > 0$ Eq. (4) has an infinite series of real roots and no more than one pair of complex-conjugate roots, and all the roots have positive real parts.

Let $\{\lambda_{ij}\}$, j = 1, 2, ..., be a series of roots of Eq. (4) arranged in order of increase of the real part. Then as $t \to \infty$ the largest contribution to the motion will be made by normal disturbances corresponding to λ^1 such that

$$\mathrm{Re}\,\lambda^1 = \min_{l \geq 2} \mathrm{Re}\,\lambda_{l1}.$$

The rate of damping of the disturbances is determined by the quantity

$$\lambda_{l1,p} = \nu R^{-2} \lambda_{l1} = \sqrt{\frac{\sigma}{\rho R^3}} \frac{\lambda_{l1}}{\alpha}.$$

We introduce the notation

$$\mu_l = \lambda_{l1}\alpha^{-1},\ \mu = \lambda^1\alpha^{-1}.$$

The quantity $\sqrt{\sigma\rho^{-1}R^{-3}}$ is assumed to be fixed while $\alpha = \sqrt{\sigma R \rho^{-1}\nu^{-2}}$ is variable, and thus $\lambda_{l1,p}$ and λ^1_p prove to be proportional to the dimensionless quantities μ_l and μ. The relation $\mu_l(\alpha)$ is investigated below.

Certain asymptotic expressions for $\mu_l(\alpha)$ were obtained in [1]. If $\alpha \to 0$ for a fixed l, then

$$\mu_l(\alpha) = l(2l+1)(l+2)2^{-1}(2l^2+4l+3)^{-1}\alpha + 3\cdot 2^{-3}l^2(2l+1)(l+$$
$$+2)^2(4l^3+8l^2+6l+3)(l-1)^{-1}(2l+5)^{-1}(2l^2+4l+3)^{-1}\alpha^3 + O(\alpha^5).$$

If $\alpha \to \infty$ for a fixed l, then

$$\mu_l(\alpha) = i[l(l-1)(l+2)]^{1/2} + (l-1)(2l+1)\alpha^{-1} -$$
$$-\sqrt{2}(i+1)[l(l-1)(l+2)]^{-1/4}\alpha^{-3/2} + O(\alpha^{-2}). \tag{6}$$

If $l \to \infty$ for a fixed α, then

$$\mu_l(\alpha) = (1/2)\alpha l + R(\alpha, l), \tag{7}$$

where $\lim_{l\to\infty} l^{-1}R(\alpha, l) = 0$ for any $\alpha > 0$. Using the asymptotic equation for $F_l(\lambda)$ presented in [1], we can show that for $\alpha < 2$ and $l > 20$

$$|R(\alpha, l)| \leqslant (1/4)\alpha l. \tag{8}$$

Equation (4) was solved numerically for $0 < \alpha < 2$, $2 < l \leq 20$ and for $0 < \alpha < 10$, $l = 2$. A graph of the function $\mu(\alpha)$ is shown in Fig. 1. It turned out that under these conditions

$$\mu = \mu_2. \tag{9}$$

The validity of Eq. (9) for $l > 20$ and $\alpha < 2$ follows from the asymptotic equation (7) and the estimate of the remainder term (8).

Now let us show that (9) is also correct for $\alpha > 2$. On the contrary let the following inequality be satisfied for certain $l > 2$ and $\alpha > 2$:

$$\operatorname{Re} \mu_l(\alpha) \leqslant \operatorname{Re} \mu_2(\alpha). \tag{10}$$

Since from the above calculations (showing that for $\alpha > 4$ the value of λ_{21} coincides with the value calculated from Eq. (6) to within 10^{-3}) and the asymptotic equation (6) with $l = 2$ it follows that $\operatorname{Re} \lambda_{21}(\alpha) < 5$, the inequality (10) signifies the existence of a solution λ of Eq. (4) such that $\operatorname{Re} \lambda < 5$ for certain $l > 2$ and $\alpha > 2$. From (4) we get

$$2|F_l(\lambda)| = |\Phi_l(\lambda, \alpha)|. \tag{11}$$

From the last identity of (5) we can conclude that

$$2|F_l(\lambda)| \leqslant 4 \sum_{q=1}^{\infty} [(k_{l+1/2,q}^2 - \operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2]^{-1/2},$$

from which, by virtue of the inequalities $k^2_{l+1/2,q} > 5 > \operatorname{Re}\lambda$ satisfied for $l > 2$, we get

$$2|F_l(\lambda)| \leqslant 4 \sum_{q=1}^{\infty} (k_{l+1/2,q}^2 - 5)^{-1} = 2F_l(5). \tag{12}$$

On the other hand, according to the maximum-modulus principle, applicable to the half-plane $\operatorname{Re}\lambda \leqslant 5$, we have

$$|\Phi_l(\lambda, \alpha)|^2 \geqslant \min_{-\infty<y<\infty} \{[(y^2 - 25 + 10\beta_l - \alpha^2\omega_l)^2 + 4y^2(5 - \beta_l)^2] \times$$
$$\times [(y^2 - 25 + 10\omega_l - \alpha^2\omega_l)^2 + 4y^2(5 - \omega_l)^2]^{-1}\} =$$

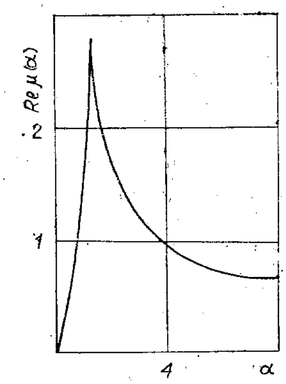

Fig. 1

$$= \min_{0<\tau<\infty} \{[(\tau - 25 + 10\beta_l - \alpha^2\omega_l)^2 + 4\tau(5-\beta_l)^2][(\tau - 25 + 10\beta_l - \alpha^2\omega_l)^2 + + 4\tau(25 - 5\omega_l - 5\beta_l + \omega_l^2) + 10(\beta_l - \omega_l)(25 - 5\omega_l - 5\beta_l + \alpha^2\omega_l)]^{-1}\}.$$

Consequently, if $\alpha^2 > 7 > 5 + 5\beta_l/\omega_l - 25/\omega_l$, then

$$|\Phi_l(\lambda,\alpha)|^2 \geqslant \min_{0<\tau<\infty} \{[(\tau - 25 + 10\beta_l - \alpha^2\omega_l)^2 + 4\tau(5-\beta_l)^2] \times \times [(\tau - 25 + 10\beta_l - \alpha^2\omega_l)^2 + 4\tau(25 - 5\omega_l - 5\beta_l + \omega_l^2)]^{-1}\} = = (5-\beta_l)^2(25 - 5\omega_l - 5\beta_l + \omega_l^2)^{-1} \geqslant (\beta_l - 5)^2\omega_l^{-2} > (l+4)^{-2}. \tag{13}$$

Calculating $F_l(5)$ directly for $l < 20$ and using the asymptotic equation [1]

$$F_l(5) \simeq \frac{1}{2}l^{-1} - \frac{3}{4}l^{-2} + \frac{7}{4}l^{-3} + O(l^{-4})$$

with an estimate of the residual term for $l > 20$, we have $2F_l(5) < (l+1)^{-1}$, and this, together with the inequalities (12) and (13), contradicts (11).

Thus, it is shown that the equality (9) is satisfied for $\alpha > \sqrt{7}$. For $2 < \alpha < \sqrt{7}$ and $l > 4$ the equality (9) follows from the fact that $\operatorname{Re} \lambda_{l_1} < 4$ and estimates of $|2F_1(\lambda)|$ and $|\Phi_1(\lambda, \alpha)|$ in the half-plane $\operatorname{Re} \lambda \leq 4$ and analogous to those made above. For $2 < \alpha < \sqrt{7}$ and $\alpha = 3$ and 4 the equality (9) is verified through direct calculations of λ_{l1}.

Thus, it is established that (9) is satisfied for all $\alpha > 0$, and the function $\mu(\alpha) = \mu_2(\alpha)$ presented in Fig. 1 actually characterizes the damping of the system at long times. Certain properties of $\mu(\alpha)$ are described below.

The function $\mu(\alpha)$ reaches a maximum at $\alpha = \alpha^* \simeq 1.305$, with $\mu(\alpha^*) \simeq 2.76$.

It is interesting that $\lambda_{21}(\alpha^*)$ is a multiple real root of Eq. (4) for $l = 2$ and $\alpha = \alpha^*$. Therefore,

$$\lim_{\alpha \to \alpha^* - 0} \mu'(\alpha) = \infty$$

and, in addition, at long times the corresponding disturbances die out as $t \exp[-\lambda_{21,p}(\alpha^*)t]$.

This situation is probably typical in the absence of rotation, when the eigenvalues are real for large enough values of the viscosity and complex for small enough values. As an illustration, we can consider the equation $\lambda^2 - 2\alpha\lambda + 1 = 0$ for which $\max_{\alpha>0} \min_{l=1,2} \operatorname{Re} \lambda_l(\alpha)$ is also reached in a multiple root.

In [1] it was shown that the system under consideration has no natural vibrational modes for $\alpha < 1.142$. The above calculations allow us to refine this number. Thus, it is established that all roots of Eqs. (11) are real if $\alpha \leq \alpha^* \simeq 1.305$. Otherwise, Eq. (4) has nonreal roots for certain values of l. If $\alpha^* < \alpha < \alpha_1 \simeq 1.611$, then Eq. (4) has one pair of complex roots for $l = 2$; for $l > 2$ all roots are real. If $\alpha_1 < \alpha < \alpha_2 \simeq 1.820$, then complex-conjugate roots exist for $l = 2$ and 3 while all roots are real for $l > 3$.

Calculations also showed that the asymptotic equation (6) well approximates $\mu_2(\alpha)$ even for $\alpha > 4$ (the deviation of the exact value of $\mu_2(\alpha)$ from the sum of the first two terms in (6) does not exceed $10^{-2}\alpha^{-1}$) and can be used in calculations involving spheres of real materials when the value of α is relatively large. For liquid metals with R = 1 cm, for example, α has the order of 10-30.

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EXPERIMENTAL INVESTIGATION OF FLOW IN SHALLOW AND DEEP CAVITIES

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In the article we present the results of an experimental investigation of the flow of an incompressible liquid in shallow and deep cavities of rectangular cross section using a laser Doppler velocity meter (LDVM). The tests were made in the laminar mode of flow in the channel ahead of the cavity. The distribution of the longitudinal and transverse velocity components in the central cross section of the cavity is obtained.

There are extremely few experimental data on the investigation of flow structure in cavities. The investigations have been confined mainly to visual observations [1]. Reports in which the static pressure and the shear stress at the cavity walls were measured are well known. The profiles of velocity and shear stress at the bottom of a shallow cavity (when the ratio of the length of the cavity to its depth is $L/H > 1.75$) were measured in [2]. It is impossible to build up a detailed concept of the character of the flow in cavities of different configurations on the basis of the available reports.

A detailed description of the experimental setup and the measurement procedure is given in [3]. Here we only provide certain information about the test section. The test cavities had the following dimensions: shallow — $L = 40$ mm, $H = 20$ mm; deep — $L = 20$ mm, $H = 40$ mm. The width of a cavity equaled the width of the plane section (100 mm). The cavities were located at a distance of 1500 mm from the plane section. During the measurements the focal region lay in a plane located at equal distances from the side walls of the cavity. The size of the focal region was $100 \times 100 \times 800$ μm. The thickness of the optical glasses was 10 mm. At a distance of 60 mm from the focal region the diameter of the laser beam was 0.5-0.6 mm. The minimum distance from the walls at which the alternate measurements of the longitudinal and transverse velocity components were made is ~1 mm. Since the optical scheme of the LDVM did not permit a determination of the direction of the velocity, flow in the cavities was investigated in detail in the case when the laminar mode of liquid motion was established in the channel and cavity. In the turbulent mode of flow we investigated only the mixing zone [the region adjacent to the upper cut of the cavity can be considered as the zone of mixing of the jet formed after separation of the stream at the point $x = 0$, $y = 0$ from the stream in the cavity (Fig. 1b)] and the boundary jet propagating along the back wall of the cavity, where the direction of motion is known.

In the case of laminar flow of liquid in the channel at $Re = 1.5 \cdot 10^3$, flow with one vortex in the upper part and a stagnant zone in the lower half was observed in the cavity with $L/H = 0.5$. After a certain time flow with two vortices rotating in opposite directions was established. The flow patterns replaced one another. With an increase in the Reynolds

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